A PENCIL OF ENRIQUES SURFACES OF INDEX ONE WITH NO SECTION

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Abstract. Monodromy arguments and deformation-and-specialization are used to prove existence of a pencil of Enriques surfaces with no section and *index 1*. The same technique completes the strategy from [GHMS05, §7.3] proving the family of *witness curves for dimension* d depends on the integer d.

1. Introduction

This paper uses monodromy and deformation-and-specialization to answer some questions related to [GHMS05]. Theorem 1.3 gives a new, elementary proof of existence of a pencil of Enriques surfaces over $\mathbb{C}$ with no section, which moreover has *index 1*. Proposition 1.4 completes the strategy from [GHMS05, §7.3] proving the family $\mathcal{H}_d$ of *witness curves* depends on the relative dimension d.

The main theorem of [GHS03] proves a rationally connected variety defined over the function field of a curve over a characteristic 0 algebraically closed field has a rational point. A converse is proved in [GHMS05]; in particular [GHMS05, Cor. 1.4] proves there is an Enriques surface without a rational point that is defined over the function field of a curve (answering a question of Serre [CS01, p. 153]). Subsequently Lafon [Laf04] gave an *explicit* pencil of Enriques surfaces defined over $\mathbb{Z}[1/2]$ whose base-change to any field of characteristic $\neq 2$ has no rational point. Hélène Esnault asked about the index of Enriques surfaces without a rational point.

Definition 1.1. Let X be a finite type scheme, algebraic space, algebraic stack, etc. over a field K. The *index* and the *minimal degree* are,

$$\begin{aligned} I(K,X) &= \gcd\{[L:K] | X(L) \neq \emptyset\}, \\ M(K,X) &= \min\{[L:K] | X(L) \neq \emptyset\}. \end{aligned}$$

Hélène Esnault asked, essentially, what is the possible index of an Enriques surface defined over a function field of a curve. In Lafon's example, $M(K, X_K) = I(K, X_K) = 2$. In [GHMS05] the index is not computed, but likely there also $I(K, X_K) > 1$.

Question 1.2 (Esnault)**.** If X is an Enriques surface defined over a function field of a curve K with no K-point, is $I(K,X) > 1$?

This has to do with whether there is an obstruction to K-points in Galois cohomology. If so and if the obstruction is compatible with restriction and corestriction,

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the order of the obstruction divides $I(K, X)$. So if there is a cohomological obstruction "explaining" non-existence of K-points, then $I(K, X_K) > 1$. The main result proves there is an Enriques surface with no K-point whose index is 1.

Theorem 1.3. *Let k be an algebraically closed field with char$(k) \neq 2, 3$ that is "sufficiently big", e.g. uncountable. There exists a flat, projective k-morphism $\pi : \mathcal{X} \to \mathbb{P}^1_k$ with the following properties,*

(i) *the geometric generic fiber of π is a smooth Enriques surface,*
(ii) *the invertible sheaf $\pi_*[\omega_\pi^{\otimes 2}]$ has degree 6,*
(iii) *for the function field K of $\mathbb{P}^1_k$ and the generic fiber X_K of π, $I(K, X_K) = 1$ and $M(K, X_K) = 3$.*

Moreover every "very general" Enriques surface over k is a fiber of such a family.

The method is simple. Over $\mathbb{P}^1$ a family of surfaces is given whose monodromy group acts as the full group of symmetries of the dual graph of the geometric generic fiber – which is the 2-skeleton of a cube. There is an action of $\mathbb{Z}/2\mathbb{Z}$ acting fiberwise, and the quotient is a pencil $\mathcal{X}/\mathbb{P}^1$ of "Enriques surfaces". The 8 vertices of the cube give a degree 4 multi-section of the pencil. The 6 faces of the cube give a degree 3 multi-section of the pencil. By monodromy considerations every multi-section of X has degree ≥ 3. The pencil X together with the degree 3 and degree 4 multi-sections deforms to a pencil whose geometric generic fiber is a smooth Enriques surface. For a general such deformation, $M(K, X_K) = 3$ and $I(K, X_K) = 1$.

The same method gives pencils of degree d hypersurfaces with minimal degree d, which is used to complete the argument from [GHMS05, Section 7.3].

Proposition 1.4. *Let B be a normal, projective variety of dimension ≥ 2 and let M be an irreducible family of irreducible curves dominating B (i.e., the morphism from the total space of the family of curves to B is dominant). There is an integer d such that M is not a* witness family for dimension d*, i.e., there is a projective, dominant morphism of relative dimension d, $\pi : \mathcal{X} \to B$, whose restriction to each curve of M has a section, but whose restriction to some smooth curve in B has no section.*

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2. The construction for hypersurfaces

Let $d, n > 0$ be integers, let k be a field, and let V be a k-vector space of dimension $n+1$. Degree d hypersurfaces in $\mathbb{P}(V)$ are parametrized by the projective space,

$$\mathbb{P}\mathrm{Sym}^d(V^\vee) = \mathrm{Proj} \bigoplus_i \mathrm{Sym}^i(\mathrm{Syt}^d(V)),$$

where $\mathrm{Syt}^d(V)$ is the vector space of symmetric tensors in $\otimes^d V$.

Let B, C be k-curves isomorphic to $\mathbb{P}^1_k$. There exists a degree d, separably-generated k-morphism $f : C \to B$ such that $\mathrm{Gal}(k(C)/k(B))$ is the full symmetric group $\mathfrak{S}_d$. This is straightforward in every characteristic – in characteristic 0 any morphism with simple branching will do.

Let $g : C \to \mathbb{P}(V^\vee)$ be a closed immersion whose image is a rational normal curve of degree n. Consider the pullback of the tautological surjection, $V \otimes_k \mathcal{O}_C \to g^*\mathcal{O}(1)$. By adjointness, there is a map $\beta : V \otimes_k \mathcal{O}_B \to f_*(g^*\mathcal{O}(1))$. For every locally free $\mathcal{O}_C$-module $\mathcal{E}$ there is the *norm sheaf* on B,

$$\mathrm{Nm}_f(\mathcal{E}) = \mathit{Hom}_{\mathcal{O}_B}(\bigwedge^d(f_*\mathcal{O}_C), \bigwedge^d(f_*\mathcal{E})),$$

together with the *norm map* of $\mathcal{O}_B$-modules,

$$\alpha'_{\mathcal{E}} : \bigotimes^d (f_*\mathcal{E}) \to \mathrm{Nm}_f(\mathcal{E}), \ \ e_1 \otimes \cdots \otimes e_d \mapsto (c_1 \wedge \cdots \wedge c_d \mapsto (c_1 \cdot e_1) \wedge \cdots \wedge (c_d \cdot e_d)),$$

for $e_1 \otimes \cdots \otimes e_d \in \bigotimes^d(f_*\mathcal{E})$ and $c_1 \wedge \cdots \wedge c_d \in \bigwedge^d(f_*\mathcal{O}_B)$. Only the restriction to the subsheaf of symmetric tensors is needed, $\alpha_{\mathcal{E}} : \mathrm{Syt}^d(f_*\mathcal{E}) \to \mathrm{Nm}_f(\mathcal{E})$. In particular, $\mathrm{Nm}_f(\mathcal{O}_C) = \mathcal{O}_B$ and $\alpha_{\mathcal{O}_C}(b \otimes \cdots \otimes b) \in \mathcal{O}_B$ is the usual norm of $b \in f_*\mathcal{O}_C$.

Denote by γ the composition,

$$\mathrm{Syt}^d(V) \otimes_k \mathcal{O}_B \xrightarrow{\mathrm{Syt}^d(\beta)} \mathrm{Syt}^d(f_*g^*\mathcal{O}(1)) \xrightarrow{\alpha_{g^*\mathcal{O}(1)}} \mathrm{Nm}_f(g^*\mathcal{O}(1)).$$

Because β is surjective, also γ is surjective. So there is an induced morphism $h : B \to \mathbb{P}\mathrm{Sym}^d(V^\vee)$. For every geometric point $b \in B$ whose fiber $f^{-1}(b)$ is a reduced set $\{c_1, \dots, c_d\}$, $h(b) = [g(c_1) \times \cdots \times g(c_d)]$. The degree of $\mathrm{Nm}_f(g^*\mathcal{O}(1))$, and thus the degree of h, is n

Denote by $\mathcal{X}_h \subset B \times \mathbb{P}(V)$ the preimage under (h, Id) of the universal hypersurface in $\mathbb{P}\mathrm{Sym}^d(V^\vee) \times \mathbb{P}(V)$, and by $\pi : \mathcal{X}_h \to B$ the projection. Let $m = \min(d, n)$ and let $S_{d,n} \subset \mathbb{Z}_{\geq 0}$ denote the additive semigroup generated by $\binom{d}{i}$ for $i = 1, \dots, m$. Denote $K = k(B)$ and denote by $\mathcal{X}_{h,K}$ the generic fiber of π.

Proposition 2.1. *Every irreducible multi-section of π has degree divisible by $\binom{d}{i}$ for $i = 1, \dots, m$. The degree of every multi-section is in $S_{d,n}$. In particular, if $d > n$ then $M(K, \mathcal{X}_{h,K}) = d$ and $I(K, \mathcal{X}_{h,K})$ is divisible by $\gcd(d, \binom{d}{2}, \dots, \binom{d}{n})$.*

Proof. Denote by $U \subset B$ the largest open subset over which f is étale and define $W = f^{-1}(U)$. For each $i = 1, \dots, m$, denote by W_i/U the relative Hilbert scheme $\mathrm{Hilb}^i_{W/U}$. Because W is étale over U, the fiber of f over a geometric point b of B is a set of d distinct points, $f^{-1}(b) = \{c_1, \dots, c_d\}$, and the fiber of $\mathrm{Hilb}^i_{W/U}$ is the set of subsets of $f^{-1}(b)$ of size i. Every geometric fiber of $\mathcal{X}_h \times_B U \to U$ is union of d hyperplanes. Denote by,

$$\mathcal{X}_h \times_B U = \mathcal{X}_h^1 \sqcup \mathcal{X}_h^2 \sqcup \cdots \sqcup \mathcal{X}_h^n,$$

the locally closed stratification where $\mathcal{X}_h^i$ is the set of points x in precisely i irreducible components of the geometric fiber $\mathcal{X}_h \otimes_{\mathcal{O}_B} \overline{\kappa}(\pi(x))$. Because every finite subset of distinct closed points on a rational normal curve over an algebraically closed field is in linearly general position, $\mathcal{X}_h^i = \emptyset$ for $i > m$; in particular every geometric fiber of $\mathcal{X}_h \times_B U \to U$ is a simple normal crossings variety. For each $i = 1, \dots, m$ the morphism $\mathcal{X}_h^i \to U$ factors as an $\mathbb{A}^{n-i}$-bundle over W_i over U. The generic point of every irreducible multi-section is contained in $\mathcal{X}_h^i$ for some $i = 1, \dots, m$. Because $\mathrm{Gal}(k(C)/k(B))$ is $\mathfrak{S}_d$, W_i is irreducible. Therefore the degree of the multi-section is divisible by $\deg(k(W_i)/k(U)) = \binom{d}{i}$. So the degree of every multi-section, irreducible or not, is in $S_{d,n}$. Moreover, the intersection of $\mathcal{X}_{h,K}$ with a general line in $\mathbb{P}(V \otimes_k K)$ is a degree d multi-section, so $M(K, \mathcal{X}_{h,K}) = d$. $\square$

Let $H_n \subset \mathrm{Hom}(B, \mathbb{P}\mathrm{Sym}^d(V^\vee))$ denote the irreducible component of morphisms of degree n. Denote by $\mathcal{X} \to H_n \times B$ the pullback by the universal morphism of the universal hypersurface in $\mathbb{P}\mathrm{Sym}^d(V^\vee) \times \mathbb{P}(V)$. For every field k' and every $[h] \in H_n(k')$, denote by $\mathcal{X}_h$ the restriction of $\mathcal{X}$ to Spec $(k') \times B$, by K' the function field $k'(B)$, and by $\mathcal{X}_{h,K'}$ the generic fiber of the projection to B.

Corollary 2.2. *Assume $d > n$. In H_n there is a countable intersection of open dense subsets such that for every $[h]$ in this set, $M(K', \mathcal{X}_{h,K'}) = d$ and $I(K', \mathcal{X}_{h,K'})$ is divisible by $gcd(d, \dots, \binom{d}{n})$. In particular this holds for the geometric generic point of H_n.*

Proof. The subset $H_n^{\mathrm{good}} \subset H_n$ where $M(K', \mathcal{X}_{K'}) \geq d$ and $\gcd(d, \dots, \binom{d}{n}) \mid I(K', \mathcal{X}_{h,K'})$ is a countable intersection of open subsets by standard Hilbert scheme arguments: the complement of this set is the union over the countably many Hilbert polynomials $P(t)$ of multi-sections of degree $< d$ or not divisible by $\gcd(d, \dots, \binom{d}{n})$ of the closed image in H_n of the relative Hilbert scheme $\mathrm{Hilb}^{P(t)}_{\mathcal{X}/H_n}$. By Proposition 2.1 H_n^{good} is nonempty, therefore it is a countable intersection of open *dense* subsets. Of course the intersection of $\mathcal{X}_{h,K'}$ with a general line in $\mathbb{P}(V \otimes_k K')$ gives a multi-section of degree d, therefore H_n^{good} is actually the set where $M(K', \mathcal{X}_{K'}) = d$ and $\gcd(d, \dots, \binom{d}{n}) \mid I(K', \mathcal{X}_{h,K'})$. ☐

2.1. **Proof of Proposition 1.4.** Let k be an uncountable, algebraically closed field. The main case of Proposition 1.4 is $B = \mathbb{P}^1_k \times \mathbb{P}^1_k$ and M is the complete linear system $|\mathcal{O}(a,b)|$. Assume first that one of $a, b = 0$, say $b = 0$. Let $f : Y \to \mathbb{P}^1_k$ be a finite, separably-generated morphism of irreducible curves of degree > 1, and let $\mathcal{X} = Y \times \mathbb{P}^1$ with projection $\pi = (f, \mathrm{Id})$. Every divisor in $|\mathcal{O}(a,0)|$ is a union of fibers of pr_1, so the restriction of π has a section. The restriction of π over every fiber of pr_2 is just f, and so has no rational section. Thus assume $a, b > 0$.

Define $n = 4ab$ and $d = n - 1$. Let V be a k-vector space of dimension $n + 1$. Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth curve in the linear system $|\mathcal{O}(1, 2b)|$. By Corollary 2.2, there exists a closed immersion of degree n, $h : C \to \mathbb{P}\mathrm{Sym}^d(V^\vee)$, such that $M(k(C), \mathcal{X}_{h,k(C)}) = d > 1$. Of course h extends to a closed immersion $j : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}\mathrm{Sym}^d(V^\vee)$ such that $j^*\mathcal{O}(1) = \mathcal{O}(2a - 1, 2b)$; after all, $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2a-1, 2b)) \to H^0(C, \mathcal{O}_C(n))$ is surjective. Define $\pi : \mathcal{X} \to \mathbb{P}^1 \times \mathbb{P}^1$ to be the base-change by j of the universal family of degree d hypersurfaces in $\mathbb{P}(V)$. By construction, the restriction over C has no section.

Every divisor in $|\mathcal{O}(a,b)|$ is a curve in $\mathbb{P}\mathrm{Sym}^d(V^\vee)$ of degree $n - b$ whose span is a linear system of hypersurfaces in $\mathbb{P}(V)$ of (projective) dimension $\leq n - b - (a - 1)(b - 1)$. Since $n - b < n$, this linear system has basepoints giving sections of the restriction of $\mathcal{X}$ to the divisor. This proves Proposition 1.4 for $B = \mathbb{P}^1 \times \mathbb{P}^1$ and $M = |\mathcal{O}(a,b)|$.

Let B be a normal, projective variety of dimension ≥ 2 and let M be an irreducible family of irreducible curves dominating B. There exists a smooth open subset $U \subset B$ whose complement has codimension ≥ 2 and a dominant morphism $g : U \to \mathbb{P}^1 \times \mathbb{P}^1$. Intersecting U with general hyperplanes, there exists an irreducible closed subset $Z \subset U$ such that $g|_Z : Z \to \mathbb{P}^1 \times \mathbb{P}^1$ is generically finite of some degree $e > 0$. For the geometric generic point of M, the intersection of the corresponding curve with U is nonempty, and the closure of the image under f is a divisor in the linear

system $|\mathcal{O}(a',b')|$ for some integers a',b'. Let $a \geq a'$, and $b \geq b'$ be integers such that $4ab > e+1$. There exists a projective, dominant morphism $\pi : \mathcal{X} \to \mathbb{P}^1 \times \mathbb{P}^1$ whose restriction over every divisor in $|\mathcal{O}(a,b)|$ has a section, but whose restriction over a general divisor in $|\mathcal{O}(1,2b)|$ has minimal degree $4ab-1$.

Define $\mathcal{X}_B \subset B \times \mathcal{X}$ to be the closure of $U \times_{\mathbb{P}^1\times\mathbb{P}^1} \mathcal{X}$. Then $\pi_B : \mathcal{X}_B \to B$ is a projective dominant morphism. For the geometric generic point of M, the restriction of π_B to the curve has a section because the restriction of π to the image in $\mathbb{P}^1 \times \mathbb{P}^1$ has a section. Let $C_B \subset Z$ be the preimage of a general curve C in $|\mathcal{O}(1,2b)|$. The morphism $C_B \to C$ has degree $e < 4ab-1$. Because every multi-section of π over C has degree $\geq 4ab-1$, π_B has no section over C_B.

3. The construction for Enriques surfaces

Let k be a field of characteristic $\neq 2, 3$, and let V_+ and V_- be 3-dimensional k-vector spaces. Denote $V = V_+ \oplus V_-$ and $V' = \mathrm{Sym}^2(V_+^\vee) \oplus \mathrm{Sym}^2(V_-^\vee)$. Denote $G = \mathrm{Grass}(3, V')$, parametrizing 3-dimensional *subspaces* of V'. This is a parameter space for Enriques surfaces. There are 2 descriptions of the universal family, each useful. First, let $\pi_Z : Z \to \mathbb{P}(V_+) \times \mathbb{P}(V_-)$ be the projective bundle of the locally free sheaf $\mathrm{pr}_+^*\mathcal{O}_{\mathbb{P}(V_+)}(-2) \oplus \mathrm{pr}_-^*\mathcal{O}_{\mathbb{P}(V_-)}(-2)$. A general complete intersection of 3 divisors in $|\mathcal{O}_Z(1)|$ is an Enriques surface. Because $H^0(Z, \mathcal{O}_Z(1)) = V'$, the parameter space for these complete intersections is G. Second, G parametrizes complete intersections in $\mathbb{P}(V)$ of 3 quadric divisors that are invariant under the involution ι of $\mathbb{P}(V)$ whose (-1)-eigenspace is V_- and whose $(+1)$-eigenspace is V_+. A general such complete intersection is a K3 surface on which ι acts as a fixed-point-free involution; the quotient by ι is an Enriques surface. The two descriptions are equivalent: the involution extends to an involution $\widetilde{\iota}$ on the blowing up $\widetilde{\mathbb{P}(V)}$ of $\mathbb{P}(V)$ along $\mathbb{P}(V_+) \cup \mathbb{P}(V_-)$ and the quotient is Z. Denote by $\mathcal{X} \to G$ the universal family of Enriques surfaces, and denote by $\mathcal{Y} \to G$ the universal family of K3 covers.

Let B, C, D be k-curves isomorphic to $\mathbb{P}^1_k$. There exists a degree 2, separably-generated morphism $g : D \to C$ and a degree 3, separably-generated morphism $f : C \to B$ such that $\mathrm{Gal}(k(D)/k(B))$ is the full wreath product $\mathfrak{W}_{3,2}$, i.e., the semidirect product $(\mathfrak{S}_2)^3 \rtimes \mathfrak{S}_3$. In characteristic 0, this holds whenever g and f have simple branching and the branch points of g are in distinct, reduced fibers of f. There is an involution ι_D of D commuting with g.

Let $j : D \to \mathbb{P}(V^\vee)$ be a closed immersion equivariant for ι_D and ι whose image is a rational normal curve of degree 5. By the construction in Section 2, there is an associated morphism $i : C \to \mathbb{P}\mathrm{Sym}^2(V^\vee)$. Because j is equivariant, i factors through $\mathbb{P}(V')$. By a straightforward computation, $i^*\mathcal{O}(1) = \mathrm{Nm}_g(j^*\mathcal{O}(1)) \cong \mathcal{O}_C(5)$. The pushforward by f_* of the pullback by i^* of the tautological surjection is a surjection $(V')^\vee \otimes \mathcal{O}_B \to f_*i^*\mathcal{O}(1)$. The sheaf $f_*i^*\mathcal{O}(1)$ is locally free, in fact $f_*i^*\mathcal{O}(1) \cong f_*\mathcal{O}_C(5) \cong \mathcal{O}_B(1)^3$, so there is an induced morphism $h : B \to G$. Denote by $\pi_h : \mathcal{X}_h \to B$ and $\rho_h : \mathcal{Y}_h \to B$ the base-change by h of $\mathcal{X}$ and $\mathcal{Y}$. Denote $K = k(B)$ and denote by $\mathcal{X}_{h,K}$ the generic fiber of π_h.

Proposition 3.1. *Every irreducible multi-section of π_h has degree divisible by 3 or 4. In particular $M(K, \mathcal{X}_{h,K}) = 3$.*

Proof. Denote by $U \subset B$ the open set over which $f \circ g$ is étale, and denote by $W \subset D$ the preimage of U. Denote by $c : \widetilde{W} \to U$ the Galois closure of W/U. Then

$c^*f_*\mathcal{O}_C|_U \cong \mathcal{O}_{\widetilde{W}}\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\}$ for idempotents $\mathbf{a}_p$, $p=1,2,3$. And $c^*g_*f_*\mathcal{O}_D|_U \cong \mathcal{O}_{\widetilde{W}}\{\mathbf{b}_{1,1},\mathbf{b}_{1,2},\mathbf{b}_{2,1},\mathbf{b}_{2,2},\mathbf{b}_{3,1},\mathbf{b}_{3,2}\}$ for idempotents $\mathbf{b}_{p,q}$, $p=1,2,3$, $q=1,2$. Of course $\mathbf{a}_p \mapsto \mathbf{b}_{p,1}+\mathbf{b}_{p,2}$, $p=1,2,3$. The action of the Galois group $\mathfrak{W}_{3,2}$ on $\mathbf{a}_p$ is by the symmetric group $\mathfrak{S}_3$, and the action on $\mathbf{b}_{p,q}$ is the standard representation of the wreath product.

For each $p=1,2,3$ and $q=1,2$, denote by $j_{p,q}:\widetilde{W}\to\mathbb{P}(V^\vee)$ the morphism obtained by composing the idempotent $\mathbf{b}_{p,q}:\widetilde{W}\to\widetilde{W}\times_U W$ with the basechange of j. In particular, $\iota\circ j_{p,1}=j_{p,2}$. Denote by $\Lambda_{p,q}\subset\widetilde{W}\times\mathbb{P}(V)$ the pullback by $(j_{p,q},\mathrm{Id})$ of the universal hyperplane. Denote by $\mathcal{Y}_{\widetilde{W}}$ the base-change to $\widetilde{W}$ of $\mathcal{Y}_h$. Then,

$$\mathcal{Y}_{\widetilde{W}} = \bigcup_{(q_1,q_2,q_3)\in\{1,2\}^3} (\Lambda_{1,q_1}\cap\Lambda_{2,q_2}\cap\Lambda_{3,q_3}).$$

There is a locally closed stratification,

$$\mathcal{Y}_{\widetilde{W}} = \mathcal{Y}^3_{\widetilde{W}} \sqcup \mathcal{Y}^4_{\widetilde{W}} \sqcup \mathcal{Y}^5_{\widetilde{W}},$$

where $\mathcal{Y}^l_{\widetilde{W}}$ is the set of points lying in the intersection of precisely l of the $\Lambda_{p,q}$. The stratum $\mathcal{Y}^3_{\widetilde{W}}$ is the union of 8 connected, open subsets,

$$\Lambda_{(q_1,q_2,q_3)} \subset (\Lambda_{1,q_1}\cap\Lambda_{2,q_2}\cap\Lambda_{3,q_3}),$$

for $q_1,q_2,q_3\in\{1,2\}$. Each connected component is a dense open subset of a $\mathbb{P}^2$-bundle over $\widetilde{W}$. The stratum $\mathcal{Y}^4_{\widetilde{W}}$ is the union of 12 connected, open subsets,

$$\begin{array}{rcl}
\Lambda_{(*,q_2,q_3)} & \subset & (\Lambda_{1,1}\cap\Lambda_{1,2})\cap\Lambda_{2,q_2}\cap\Lambda_{3,q_3},\\
\Lambda_{(q_1,*,q_3)} & \subset & \Lambda_{1,q_1}\cap(\Lambda_{2,1}\cap\Lambda_{2,2})\cap\Lambda_{3,q_3},\\
\Lambda_{(q_1,q_2,*)} & \subset & \Lambda_{1,q_1}\cap\Lambda_{2,q_2}\cap(\Lambda_{3,1}\cap\Lambda_{3,2})
\end{array}$$

for $q_1,q_2,q_3\in\{1,2\}$. Each connected component is a dense open subset of a $\mathbb{P}^1$-bundle over $\widetilde{W}$. Finally $\mathcal{Y}^5_{\widetilde{W}}$ is the union of 6 connected sets,

$$\begin{array}{rcl}
\Lambda_{(*,*,q_3)} & = & (\Lambda_{1,1}\cap\Lambda_{1,2})\cap(\Lambda_{2,1}\cap\Lambda_{2,2})\cap\Lambda_{3,q_3},\\
\Lambda_{(*,q_2,*)} & = & (\Lambda_{1,1}\cap\Lambda_{1,2})\cap\Lambda_{2,q_2}\cap(\Lambda_{3,1}\cap\Lambda_{3,2}),\\
\Lambda_{(q_1,*,*)} & = & \Lambda_{1,q_1}\cap(\Lambda_{2,1}\cap\Lambda_{2,2})\cap(\Lambda_{3,1}\cap\Lambda_{3,2})
\end{array}$$

for $q_1,q_2,q_3\in\{1,2\}$. Each connected component projects isomorphically to $\widetilde{W}$.

There is a bijection between multi-sections of $\mathcal{Y}_h$ over U and Galois invariant multi-sections of $\mathcal{Y}_{\widetilde{W}}$ over $\widetilde{W}$. An irreducible multi-section of $\mathcal{Y}_h$ determines a multi-section of $\mathcal{Y}_{\widetilde{W}}$ contained in a single stratum $\mathcal{Y}^l_{\widetilde{W}}$. The action of the Galois group $\mathfrak{W}_{3,2}$ on the connected components of $\mathcal{Y}^l_{\widetilde{W}}$ is the obvious one; in particular, it acts transitively on the set of connected components. So every Galois invariant multi-section in $\mathcal{Y}^3_{\widetilde{W}}$ has degree divisible by 8, every Galois invariant multi-section in $\mathcal{Y}^4_{\widetilde{W}}$ has degree divisible by 12, and every Galois invariant multi-section in $\mathcal{Y}^5_{\widetilde{W}}$ has degree divisible by 6. Therefore every irreducible multi-section of $\mathcal{Y}_h$ has degree divisible by 8 or 6. Because $\mathcal{Y}_h$ is a double-cover of $\mathcal{X}_h$, every irreducible multi-section of $\mathcal{X}_h$ has degree divisible by 4 or 3. In particular, the minimal degree of a multi-section of $\mathcal{X}_h$ is 3. □

Because $f_*i^*\mathcal{O}(1) \cong \mathcal{O}_B(1)^3$, the scheme $\mathcal{X}_h \subset B \times Z$ is a complete intersection of 3 divisors in the linear system $|\mathrm{pr}_B^*\mathcal{O}_B(1) \otimes \mathrm{pr}_Z^*\mathcal{O}_Z(1)|$. A general deformation of this complete intersection is a pencil of Enriques surfaces satisfying Theorem 1.3 (i) and (ii) with $M(K, X_K) \geq 3$, $I(K, X_K) \mid 4$ (this is valid so long as $\mathrm{char}(k) \neq 2$). For (iii), it is necessary to deform the pencil together with the degree 3 multi-section. This requires a bit more work, and the hypothesis $\mathrm{char}(k) \neq 2, 3$.

The stratum $\mathcal{Y}^5_{\widetilde{W}}$ is Galois invariant and determines a degree 3 multi-section of $\mathcal{X}_h$. As a $\mathfrak{W}_{3,2}$-equivariant morphism to $\widetilde{W}$, $\mathcal{Y}^5_{\widetilde{W}}$ is just the base-change of D, and the morphism $\mathcal{Y}^5_{\widetilde{W}} \to \mathbb{P}(V)$ is Galois invariant. By étale descent it is the base-change of a morphism $j' : D \to \mathbb{P}(V)$. Now j' induces a morphism to $\widetilde{\mathbb{P}(V)}$, the blowing up of $\mathbb{P}(V)$ along $\mathbb{P}(V_+) \cup \mathbb{P}(V_-)$. Because j' is equivariant for ι and ι_D, the quotient morphism $D \to Z$ factors through C, i.e., there is an induced morphism $i' : C \to Z$. By a straightforward enumerative geometry computation, j' has degree 5 with respect to $\mathcal{O}_{\mathbb{P}(V)}(1)$. Therefore i' has degree 5 with respect to $\mathcal{O}_Z(1)$. The degree 3 multi-section of $\mathcal{X}_h$ is the image of $(f, i') : C \to B \times Z$.

Lemma 3.2. *If f, g and j are general, then $(i')^* : H^0(Z, \mathcal{O}_Z(1)) \to H^0(C, \mathcal{O}_C(5))$ is surjective.*

Proof. The condition that $(i')^*$ is surjective is an open condition in families, hence it suffices to verify $(i')^*$ is surjective for a single choice of f, g and j – even one for which $\mathrm{Gal}(k(D)/k(B))$ is not $\mathfrak{W}_{3,2}$. Choose homogeneous coordinates $[S_0, S_1]$ on D, $[T_0, T_1]$ on C and $[U_0, U_1]$ on B. Define $g([S_0, S_1]) = [S_0^2, S_1^2]$ and $f([T_0, T_1]) = [T_0^3, T_1^3]$. Denote by μ_6 the group scheme of 6^{th} roots of unity. There is an action of μ_6 on D by $\zeta \cdot [S_0, S_1] = [S_0, \zeta S_1]$. This identifies μ_6 with $\mathrm{Gal}(k(D)/k(B))$.

Let $\mathbf{e}_{+,0}, \mathbf{e}_{+,1}, \mathbf{e}_{+,2}$ and $\mathbf{e}_{-,0}, \mathbf{e}_{-,1}, \mathbf{e}_{-,2}$ be ordered bases of V_+ and V_- respectively, and let $X_{+,0}, X_{+,1}, X_{+,2}$ and $X_{-,0}, X_{-,1}, X_{-,2}$ be the dual ordered bases of $V_+^\vee$ and $V_-^\vee$ respectively. There is an action of μ_6 on V by,

$$\zeta \cdot [X_{+,0}, X_{+,1}, X_{+,2}, X_{-,0}, X_{-,1}, X_{-,2}] = \\ \left[X_{+,0}, \zeta^2 X_{+,1}, \zeta^4 X_{+,2}, \zeta X_{-,0}, \zeta^3 X_{-,1}, \zeta^5 X_{-,2}\right]$$

and a dual action on $V^\vee$. Define $j : D \to \mathbb{P}(V)$ with respect to the ordered basis $\mathbf{e}_{+,0}, \ldots, \mathbf{e}_{-,2}$, to be the μ_6-equivariant morphism,

$$j([S_0, S_1]) = [S_0^5, S_0^3 S_1^2, S_0 S_1^3, S_0^4 S_1, S_0^2 S_1^3, S_1^5].$$

In this case $U = D_+(U_0 U_1) \subset B$ and $\widetilde{W} = W = D_+(S_0 S_1) \subset C$. It is straightforward to compute j' with respect to the dual ordered basis $X_{+,0}, \ldots, X_{-,2}$,

$$j'([S_0, S_1]) = [S_1^5, S_0^2 S_1^3, S_0^4 S_1, S_0 S_1^4, S_0 S_1^4, S_0^3 S_1^2, S_0^5].$$

As a double-check, observe this is μ_6-equivariant. The induced map $(j')^*$ is,

$$\begin{array}{rclrclrcl}
X_{+,0}X_{+,0} & \mapsto & T_1^5, & X_{+,0}X_{+,1} & \mapsto & T_0 T_1^4, & X_{+,0}X_{+,2} & \mapsto & T_0^2 T_1^3, \\
X_{+,1}X_{+,1} & \mapsto & T_0^2 T_1^3, & X_{+,1}X_{+,2} & \mapsto & T_0^3 T_1^2, & X_{+,2}X_{+,2} & \mapsto & T_0^4 T_1, \\
X_{-,0}X_{-,0} & \mapsto & T_0 T_1^4, & X_{-,0}X_{-,1} & \mapsto & T_0^2 T_1^3, & X_{-,0}X_{-,2} & \mapsto & T_0^3 T_1^2, \\
X_{-,1}X_{-,1} & \mapsto & T_0^3 T_1^2, & X_{-,1}X_{-,2} & \mapsto & T_0^4 T_1, & X_{-,2}X_{-,2} & \mapsto & T_0^5.
\end{array}$$

This is surjective by inspection. $\square$

3.1. **Proof of Theorem 1.3.** The subvariety $\mathcal{X}_h \subset B \times Z$ is a complete intersection of 3 divisors in the linear system $|\mathrm{pr}_B^*\mathcal{O}_B(1) \otimes \mathrm{pr}_Z^*\mathcal{O}_Z(1)|$, each containing $(f, i')(C)$. Denote by $\mathcal{I}$ the ideal sheaf of $(f, i')(C) \subset B \times Z$, and denote $I = H^0(B \times Z, \mathcal{I} \otimes \mathrm{pr}_B^*\mathcal{O}_B(1) \otimes \mathrm{pr}_Z^*\mathcal{O}_Z(1))$. The projective space of I is the linear system of divisors on $B \times Z$ in the linear system $|\mathrm{pr}_B^*\mathcal{O}_B(1) \otimes \mathrm{pr}_Z^*\mathcal{O}_Z(1)|$ that contain $(f, i')(C)$. The Grassmannian $G' = \mathrm{Grass}(3, I)$ is the parameter space for deformations of $\mathcal{X}_h$ that contain $(f, i')(C)$. For the same reason as in Corollary 2.2, in G' there is a countable intersection of dense open subsets parametrizing subvarieties $\mathcal{X}' \subset B \times Z$ with $M(K, \mathcal{X}'_K) \geq 3$ and $I(K, \mathcal{X}'_K) \mid 4$. By construction, $\mathcal{X}'$ contains the degree 3 multi-section $(f, i')(C)$. Therefore $M(K, \mathcal{X}'_K) = 3$ and $I(K, \mathcal{X}'_K) = 1$. It is straightforward to compute $\mathrm{pr}_B * [\omega_{\mathcal{X}'/B}^{\otimes 2}] \cong \mathcal{O}_B(6)$. So to prove the theorem, it suffices to prove every "very general" Enriques surface occurs as a fiber of some $\mathcal{X}'$, i.e., for a general $[X] \in G$, X occurs as $\mathrm{pr}_Z(\mathcal{X}' \cap \pi_B^{-1}(b))$ for some choice of f, g, i and $b \in B$.

A general 0-dimensional, length 3 subscheme of Z occurs as $i'(f^{-1}(b))$ for some choice of f, g, i and $b \in B$. So for a general Enriques surface $[X] \in G$ and a general choice of 0-dimensional, length 3 subscheme of X, X is a complete intersection of 3 divisors in the linear system $|\mathcal{O}_Z(1)|$ containing $i'(f^{-1}(b))$ for some choice of f, g, i and b. To prove that a general $[X] \in G$ is the fiber over b of $\mathcal{X}'$ for some f, g, i and $[\mathcal{X}'] \in G'$, it suffices to prove every divisor in the linear system $|\mathcal{O}_Z(1)|$ containing $i'(f^{-1}(b))$ is the fiber over b of a divisor in the linear system $|\mathcal{I} \otimes \mathcal{O}_B(1) \otimes \mathcal{O}_Z(1)|$.

There is a short exact sequence,

$$0 \to \mathcal{I} \otimes \mathrm{pr}_Z^*\mathcal{O}_Z(1) \to \mathrm{pr}_Z^*\mathcal{O}_Z(1) \to \mathrm{pr}_Z^*\mathcal{O}_Z(1)|_C \to 0,$$

giving a short exact sequence,

$$0 \to \mathrm{pr}_{B,*}(\mathcal{I} \otimes \mathrm{pr}_Z^*\mathcal{O}_Z(1)) \to \mathrm{pr}_{B,*}\mathrm{pr}_Z^*\mathcal{O}_Z(1) \to \mathrm{pr}_{B,*}(\mathrm{pr}_Z^*\mathcal{O}_Z(1)|_C) \to 0.$$

Because $(i')^*$ is surjective, $\mathrm{pr}_{B,*}(\mathcal{I} \otimes \mathrm{pr}_Z^*\mathcal{O}_Z(1))$ is a locally free sheaf with $h^1 = 0$. So it is $\cong \mathcal{O}_B^6 \oplus \mathcal{O}_B(-1)^3$. Twisting by $\mathcal{O}_B(1)$, $\mathrm{pr}_{B,*}(\mathcal{I} \otimes \mathrm{pr}_B^*\mathcal{O}_B(1) \otimes \mathrm{pr}_Z^*\mathcal{O}_Z(1))$ is generated by global sections. Therefore every divisor on Z in the linear system $|\mathcal{O}_Z(1)|$ containing the scheme $i'(f^{-1}(b))$ is the fiber over b of a divisor on $B \times Z$ in the linear system $|\mathcal{I} \otimes \mathrm{pr}_B^*\mathcal{O}_B(1) \otimes \mathrm{pr}_Z^*\mathcal{O}_Z(1)|$.

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